HIGHER GENUS CURVES ON TORIC VARIETIES

MIHAI HALIC

Introduction

This article is intended to be an application to the case of toric varieties of the ideas developed in my paper [8], where I have studied the problem of computing the Gromov-Witten invariants of quotient varieties.

The quantum cohomology ring of toric varieties was studied in [3], where is given an explicit formula for the quantum multiplication on Fano toric varieties, and also in [11], where the virtual localization technique is used in order to compute genus zero invariants of toric varieties. For the higher genus invariants, the combinatorics which appears makes the application of the virtual localization formula difficult.

The aim of this article is less ambitious, we just want to have an as neat as possible description of the space of morphisms from a fixed curve into a smooth and projective toric variety, provided the degree of these morphisms is sufficiently large. We impose the restriction on the degree because it turns out that in this case the space of morphisms in question is smooth and has the expected dimension. Eventually, we will compute intersection numbers on a certain natural smooth and projective compactification of it. As a disclaimer, we should say from the very beginning that we do not pretend to describe the stable map compactification à la Kontsevich-Manin, which is hard to grasp, but instead what we find is birational to an irreducible component of this space of stable maps.

The article begins with an introductory section whose purpose is that of fixing the notations and recalling generalities on toric varieties. The study actually starts with the second section which treats the problem of compactifying the space of morphisms from a smooth and projective curve into a smooth and projective toric variety; the conclusions in this direction are contained in proposition 2.2 and corollary 2.4. After the description of the generators of the cohomology ring and the relations among them, we apply in the last section the localization method for computing intersection products on our space of morphisms. The general formula given in theorem 4.5, which is the main result of the present article, has the shortcoming of being too combinatorial and therefore not explicit enough. Neverthless, using it we are able to give in proposition 4.7 explicit formulae for certain intersection products on one hand and, on the other hand, to derive in proposition 4.8 vanishings induced by the primitive collections of the fan defining the toric variety we start with.

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1. Setting up the problem

Toric varieties are studied in great detail in [6, 7, 10], but I shall prefer the very nice synthesis of the topic which can be found in the third chapter of [5]. We are starting with the notations: $X = X_\Sigma$ stands for a smooth and projective toric variety defined by the fan $\Sigma \subset M^\vee_{\mathbb{R}} := \mathsf{Hom}_{\mathbb{Z}}(M, \mathbb{R})$, with $M \cong \mathbb{Z}^n$, whose one dimensional faces are denoted $\Sigma(1)$; we let $r := \#\Sigma(1)$ and $l := r - n$. The assumptions on X imply that the integral generators of any n-dimensional cone of Σ form a $\mathbb{Z}$-basis of $M^\vee$. We denote $e^1, \ldots, e^l, e^{l+1}, \ldots, e^r$ the integral generators of $\Sigma(1)$, and we are further assuming that $(e^{l+1}, \ldots, e^r)$ actually generate an n-dimensional cone. Since they form a $\mathbb{Z}$-basis of $M^\vee$, there are integers a^λ_ν such that

$$e^\lambda + a^\lambda_\nu e^{l+\nu} = 0. \tag{1.1}$$

Here we are using the usual summing convention over the indices; in the whole paper we let $\lambda \in \{1, \ldots, l\}$, $\nu \in \{1, \ldots, n\}$ and $\rho \in \{1, \ldots, r\}$. We have the exact sequence of tori

$$1 \longrightarrow T \xrightarrow{\varepsilon} (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow S \longrightarrow 1, \tag{1.2}$$

where $T := \mathsf{Hom}_{\mathbb{Z}}(A^1(X); \mathbb{C}^*)$ and $S := M^\vee \otimes_{\mathbb{Z}} \mathbb{C}^*$, which in the basis $(e^{l+1}, \ldots, e^r)$ takes the form

$$1 \longrightarrow T \xrightarrow{\varepsilon} (\mathbb{C}^*)^r \longrightarrow S \longrightarrow 1, \tag{1.3}$$

with the homomorphism ε defined by the characters $(\chi_\rho)_\rho$ as follows:

$$\chi_\lambda(t) = t_\lambda \quad \text{and} \quad \chi_{l+\nu}(t) = t^{\underline{a}_\nu} = t_1^{a^1_\nu} \cdot \ldots \cdot t_l^{a^l_\nu}. \tag{1.4}$$

We should keep in mind that this last description depends on the choice of a n-dimensional cone of Σ, and this remark will be used over and over in the paper. The homomorphism ε induces a T-action on $\mathbb{C}^r$, and X is simply the quotient for this action. More precisely, there is a T-invariant open subset $\Omega \subset \mathbb{C}^r$ whose complement $Z_X := \mathbb{C}^r \setminus \Omega$ has codimension at least two, such that $X = \Omega/T$. In fact Z_X is a union of linear subspaces of $\mathbb{C}^r$,

$$Z_X = \bigcup_\pi \mathbb{A}(\pi), \tag{1.5}$$

where π runs over the set of so-called primitive collections of Σ (see sections 1 and 2 of [4] for definition and proof).

The generators $(e^\rho)_\rho$ of $\Sigma(1)$ define respectively the divisors $(D_\rho)_\rho$ on X which, as elements of $A^1(X)$, satisfy the linear relations

$$D_{l+\nu} - a^\lambda_\nu D_\lambda = 0, \text{ for all } \nu = 1, \ldots, n. \tag{1.6}$$

In this way we get an isomorphism $A^1(X) \cong \mathbb{Z}D_1 \oplus \cdots \oplus \mathbb{Z}D_l$, determined by the choice of the n-dimensional cone $(e^{l+1}, \ldots, e^r)$ of Σ.

Definition 1.1. For C a smooth and projective curve of genus g, we say that a morphism $u : C \to X$ has multi-degree $\underline{d} = (d_\rho)_\rho$ if $d_\rho = D_\rho \cdot u_*C$ for $\rho = 1, \ldots, r$.

Of course, in definition above the integers d_ρ are not independent but are related by $d_{l+\nu} = a^\lambda_\nu d_\lambda$ for all ν.

The following well-known 'Euler sequence' on X,

$$0 \longrightarrow \mathcal{O}_X^{\oplus l} \longrightarrow \bigoplus_\rho \mathcal{O}(D_\rho) \longrightarrow \mathsf{T}_X \longrightarrow 0, \tag{1.7}$$

and the Riemann-Roch theorem immediately implies the

Lemma 1.2. *The space $\mathsf{Mor}_{\underline{d}}(C,X)$ of morphisms from C to X having multi-degree $\underline{d}$ is smooth as soon as $d_\rho > 2g-1$ for all $\rho = 1,\dots,r$. In this case, it is also irreducible and has the expected dimension*

$$\dim \mathsf{Mor}_{\underline{d}}(C,X) = \sum_\rho d_\rho - n(g-1). \tag{1.8}$$

Proof. All the statements are obvious except the one concerning the irreducibility of $\mathsf{Mor}_{\underline{d}}(C,X)$ which will be proved in corollary 2.4. □

As this first section is devoted to set up the stage, let us mention a probably well-known generality about toric varieties which will be needed in the proof of proposition 2.2.

Proposition 1.3. *Let X be a smooth and projective toric variety as before. There is an ample line bundle $A \to \mathbb{C}^r$ having the properties:*
(i) *it linearizes the standard $(\mathbb{C}^*)^r$-action on $\mathbb{C}^r$;*
(i) *the corresponding set of T-semi-stable points is precisely $\Omega \subset \mathbb{C}^r$.*

Proof. Since X is projective, there are characters β of T such that the associated line bundles $A_\beta := \Omega \times_\beta \mathbb{C} \to X$ are very ample. Sections of A_β can be naturally identified with T-equivariant holomorphic functions on Ω, and *a fortiori* on $\mathbb{C}^r$ because $Z_X = \mathbb{C}^r \setminus \Omega$ has codimension at least two. We say that $f : \mathbb{C}^r \to \mathbb{C}$ is T-equivariant if

$$f\big(\varepsilon(t') \times z\big) = \beta(t')f(z),\ \forall t' \in T \text{ and } \forall z \in \mathbb{C}^r,$$

and denote $\mathcal{O}(\mathbb{C}^r)_\beta$ the (finite dimensional) vector space of such functions. There is a natural map

$$\begin{aligned} F : \mathbb{C}^r &\longrightarrow \mathcal{O}(\mathbb{C}^r)_\beta{}^\vee, \\ z &\longmapsto F(z) : \quad \langle F(z), f\rangle = f(z), \end{aligned}$$

having the properties:
- it covers the projective embedding of X defined by the linear system of A_β;
- $F(z) = 0$ for all $z \in Z_X$ and $F(z) \neq 0$ for all $z \in \Omega$;
- $F\big(\varepsilon(t') \times z\big) = \beta(t')F(z)$ for all $t' \in T$ and $z \in \mathbb{C}^r$.

In a certain basis of $\mathcal{O}(\mathbb{C}^r)_\beta{}^\vee$, the $(\mathbb{C}^*)^r$-action can be diagonalized, and we denote $\mu_1,\dots,\mu_\omega$ the corresponding characters; they have the property that $\mu_k \circ \varepsilon = \beta$ for $k = 1,\dots,\omega$. With respect to this basis, we are defining the $(\mathbb{C}^*)^r$-action on $\mathbb{P}\big(\mathcal{O}(\mathbb{C}^r)_\beta{}^\vee \oplus \mathbb{C}\big)$, together with a linearization in the standard ample line bundle over this projective space as follows: $(\mathbb{C}^*)^r$ acts on $\mathcal{O}(\mathbb{C}^r)_\beta{}^\vee$ by the characters $1, \mu_1^{-1}\mu_2, \dots, \mu_1^{-1}\mu_\omega$ and on the $\mathbb{C}$-term by μ_1^{-1}. Similarly, we let $(\mathbb{C}^*)^r$ act on $\mathbb{C}^{r+1} = \mathbb{C}^r \oplus \mathbb{C}$, in the standard fashion on $\mathbb{C}^r$ and trivially on the last component.

The ample line bundle $A \to \mathbb{C}^r$ we are looking for is obtained by restricting via

$$\mathbb{C}^r \cong \mathrm{Graph}(F) \subset \mathbb{P}\big(\mathbb{C}^r \oplus \mathbb{C}\big) \times \mathbb{P}\big(\mathcal{O}(\mathbb{C}^r)_\beta{}^\vee \oplus \mathbb{C}\big)$$

the natural (tensor product) ample line bundle on the product of these projective spaces. After replacing β with a sufficiently large power of it, one may check that the points $(z,1) \times (0,1)$, $z \in \mathbb{C}^r$, are T-unstable, while the points $(z,1) \times (y,1)$,

$y \neq 0$, are T-semi-stable. Consequently the T-semi-stable set of $\mathbb{C}^r$ is precisely Ω. $\square$

We continue the preparatory material by fixing once for all a point $\zeta_0 \in C$ and we consider the Poincaré bundle $\mathcal{L}_0 \to \mathcal{J} \times C$ whose restriction $\mathcal{L}_0|_{\mathcal{J}\times\{\zeta_0\}} = \mathcal{O}_{\mathcal{J}}$; for $d \in \mathbb{Z}$, we define $\mathcal{L}_d := \mathcal{L}_0 \otimes \mathrm{pr}_C^* \mathcal{O}(d\zeta_0)$. As one expects, $\mathcal{J}$ denotes the Jacobian variety of C and, for all integers d, $\mathcal{L}_d$ parameterizes line bundles of degree d over the curve C.

The topological type of a holomorphic principal T-bundle over C is determined by its multi-degree $\underline{d}' = (d_1, \ldots, d_l)$; holomorphic principal T-bundles over C with fixed multi-degree, are parameterized by the l^{th} power of the Jacobian of C. We denote

$$\mathcal{P}_{\underline{d}'} \longrightarrow \mathcal{J}^l \times C$$

the universal principal T-bundle parameterizing principal T-bundles over C with multi-degree $\underline{d}'$, trivialized at ζ_0. The bundle $\mathcal{P}_{\underline{d}'}$ is uniquely determined by the choice of the Poincaré bundles $\mathcal{L}_d$ above.

Later on we will see that is useful to consider the principal bundle

$$\mathcal{P}_{\underline{d}} \longrightarrow \mathcal{J}^r \times C$$

which parameterizes $(\mathbb{C}^*)^r$-bundles over C having multi-degree $\underline{d} = (d_1, \ldots, d_r)$. For

$$\begin{aligned} &\psi : \mathcal{J}^l \longrightarrow \mathcal{J}^r \quad \text{defined by} \\ &(L_1, \ldots, L_l) \longmapsto (L_1, \ldots, L_l, a_1^\lambda L_\lambda, \ldots, a_n^\lambda L_\lambda), \end{aligned} \tag{1.9}$$

we get the commutative diagram

$$\begin{tikzcd} \mathcal{P}_{\underline{d}'} \times_\varepsilon (\mathbb{C}^*)^r = \psi^* \mathcal{P}_{\underline{d}} \arrow[r] \arrow[d] & \mathcal{P}_{\underline{d}} \arrow[d] \\ \mathcal{J}^l \times C \arrow[r, "\psi"] & \mathcal{J}^r \times C. \end{tikzcd} \tag{1.10}$$

Shortly, the reason for introducing this new ingredient is that for writing the left-hand-side of (1.10) we have chosen a cone of Σ, while the right-hand-side is symmetric, the information on the structure of Σ being encoded in the map ψ.

2. Description of the space of morphisms

All the subsequent constructions are motivated by the following very simple remark: given a morphism $u : C \to X$ having multi-degree $\underline{d}$, the pull-back $P := u^*\Omega \to C$ is a holomorphic principal T-bundle whose multi-degree is $\underline{d}'$ (this is easy to see). The morphism $C = P/T \to P \times_T \mathbb{C}^r$ is just a section of a rank r vector bundle over C in which the torus T still acts, covering the identity of C. Any two sections which are in the same T-orbit give rise to the same morphism from C into X (some care is actually required at this point). This is the sort of correspondence which will be exploited in this section. The inequalities $d_\rho > 2g - 1$ appearing in lemma 1.2 will be assumed in the rest of the paper.

We start with the vector bundles associated to (1.10)

$$\begin{tikzcd} \mathcal{P}_{\underline{d}'} \times_T \mathbb{C}^r = \psi^*(\mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \mathbb{C}^r) \arrow[r] \arrow[d] & \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \mathbb{C}^r = \oplus_\rho \mathcal{L}_\rho \arrow[d] \\ \mathcal{J}^l \times C \arrow[r, "\psi"] & \mathcal{J}^r \times C, \end{tikzcd} \tag{2.1}$$

and notice that $\psi^*\mathcal{L}_\rho = \mathcal{P}_{\underline{d}'} \times_{\chi_\rho} \mathbb{C} =: \mathscr{L}_\rho$ (the characters χ_ρ are defined by (1.4)). Taking the direct images

$$\begin{tikzcd} \mathscr{V} = \oplus_\rho \mathscr{V}_\rho := p_*(\oplus_\rho \mathscr{L}_\rho) = \psi^*\mathscr{W} \arrow[r] \arrow[d] & \mathscr{W} = \oplus_\rho \mathscr{W}_\rho := p_*(\oplus_\rho \mathcal{L}_\rho) \arrow[d] \\ \mathcal{J}^l \arrow[r, "\psi"] & \mathcal{J}^r, \end{tikzcd} \tag{2.2}$$

we recognize in $\mathscr{W}_\rho$ the Picard vector bundles associated respectively to the Poincaré bundles $\mathcal{L}_\rho$; the rank of $\mathscr{W}$ is given by the formula

$$\operatorname{rk}\mathscr{W} = \sum_\rho d_\rho - r(g-1).$$

The action of T on $\mathbb{C}^r$ induces actions on $\oplus_\rho \mathscr{L}_\rho$ and $\oplus_\rho \mathcal{L}_\rho$ covering respectively the identities of $\mathcal{J}^l \times C$ and $\mathcal{J}^r \times C$ and, *a fortiori*, there are natural T-actions on $\mathscr{V}$ and $\mathscr{W}$ which cover respectively the identities of $\mathcal{J}^l$ and $\mathcal{J}^r$ and moreover preserve the decompositions $\mathscr{V} = \oplus_\rho \mathscr{V}_\rho$ and $\mathscr{W} = \oplus_\rho \mathscr{W}_\rho$.

The remark at the beginning of this section tells that the space of morphisms from C to X should be the quotient '$\mathscr{V}/T$'. Of course, this should not be taken *ad litteram* but in the spirit of geometric invariant theory. What we shall actually construct is the quotient of $\mathscr{W}$ for the T-action, and $\mathscr{V}/\!\!/T$ will be just its pull-back by ψ.

One can spot at the first glance a 'nice' Zariski open subset of $\mathscr{W}$ on which T acts freely

$$\mathscr{W}^o := \{s \in \mathscr{W} \mid \operatorname{Image} s \not\subset \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} Z_X\}. \tag{2.3}$$

As the closed subvariety $Z_X \subset \mathbb{C}^r$ which had to be 'thrown away' for obtaining X was a union of coordinate subspaces,

$$Z_W := \Gamma\big(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} Z_X\big) = \bigcup_\pi \Gamma\big(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \mathbb{A}(\pi)\big) \tag{2.4}$$

is still a union of subvector bundles of $\mathscr{W}$, and $\mathscr{W}^o = \mathscr{W} \setminus Z_W$. Even if T is acting freely on $\mathscr{W}^o$, it is possibly not so clear that the quotient $\mathscr{W}^o/T$ exists as a complex manifold.

Lemma 2.1. *$\mathscr{W}^o/T$ has a natural structure of a Hausdorff complex analytic variety.*

Proof. As expected, we put on $\mathscr{W}^o/T$ the final topology for the projection $\mathscr{W}^o \to \mathscr{W}^o/T$; we must prove that the quotient is Hausdorff when we consider on $\mathscr{W}^o$ the analytic topology given by small balls. For two sections $s, s' \in \mathscr{W}^o$ lying over the same point in $\mathcal{J}^r$, which are not in the same T-orbit, we want to prove that there are neighborhoods $\mathcal{U} \ni s$ and $\mathcal{U}' \ni s'$ such that $T\mathcal{U} \cap T\mathcal{U}' = \emptyset$ (when s and s' lie above two different points of $\mathcal{J}^r$ everything is clear). From the very definition of $\mathscr{W}^o$ we deduce that

$$s_\zeta, s'_\zeta \in \Omega, \quad \forall \zeta \in C^o,$$

with $C \setminus C^o$ a finite set. We are distinguishing two cases:

Case (1) When there is a point $\zeta \in C^o$ such that s_ζ and s'_ζ are not in the same T-orbit in Ω, the existence of the two disjoint T-invariant neighborhoods of s and s' is immediate;

Case (2) It might happen that for any $\zeta \in C^o$, the evaluations s_ζ and s'_ζ are in the same T-orbit (such a situation does appear in the simple case when we projectivize the space of sections of a line bundle). In this case, since T acts freely on Ω, there is a morphism $\tau : C^o \to T$ such that $s'_\zeta = \tau_\zeta \cdot s_\zeta$. If there are no neighborhoods $\mathcal{U}$ and $\mathcal{U}'$ as wanted, we deduce the existence of sequences $(s_k)_k \subset \mathscr{W}^o$ and $(t_k)_k \subset T$ such that $s_k \xrightarrow{\|\cdot\|} s$ and $t_k \cdot s_k \xrightarrow{\|\cdot\|} s'$, where the norm $\|\cdot\|$ on $\mathscr{W}$ is defined by

$$\|s\| := \max_{\zeta \in C} \|s_\zeta\|.$$

Let us fix a closed disk $\bar{\Delta} \in C^o$. For positive ϵ, there is a rank k_ϵ such that for $k \geq k_\epsilon$,

$$s_{k,\zeta} \in \Omega,\ \forall \zeta \in \bar{\Delta} \quad \text{and} \quad \|s'_\zeta - t_k \cdot s_k\| < \epsilon.$$

It follows that $|\tau_\zeta \cdot s_\zeta - t_k \cdot s_{k,\zeta}| < \epsilon,\ \forall \zeta \in \bar{\Delta}$, which in turn implies that $|s_\zeta - \tau_\zeta^{-1} t_k \cdot s_{k,\zeta}| < \epsilon$ for a possibly different choice of ϵ (this is because we have restricted ourselves to the compact $\bar{\Delta}$). Since for $\zeta \in \bar{\Delta}$ the s_k's take values in Ω, we deduce that $t_k \xrightarrow{k \to \infty} \tau_\zeta$ for all $\zeta \in \bar{\Delta}$; but this means that $\tau : C^o \to T$ is constant when restricted to Δ and consequently is constant everywhere. We conclude that s and s' are in the same T-orbit in $\mathscr{W}$, which contradicts our assumption. □

This lemma solves the problem of the T-action on $\mathscr{W}^o$, but one would like to work with compact manifolds, and no one guarantees that $\mathscr{W}^o/T$ is so. The purpose of the next proposition is to prove that $\mathscr{W}^o/T$ is actually a projective, and not just an analytic variety.

Proposition 2.2. *The T-action on $\mathscr{W}$ can be linearized in an ample line bundle over it, such that the semi-stable set for this action coincides with $\mathscr{W}^o$ defined by* (2.3). *Moreover, the invariant quotient $W := \mathscr{W}/\!/T$ is a smooth and projective variety of dimension*

$$\dim W = \sum_\rho d_\rho + n.$$

Proof. According to proposition 1.3 there is an ample line bundle on $A \to \mathbb{C}^r$ which linearizes the $(\mathbb{C}^*)^r$-action on $\mathbb{C}^r$, such that the corresponding T-semi-stable set is Ω, and consequently $X = \mathbb{C}^r/\!/T$. This one induces the relatively ample line bundle

$$\bar{A} := \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} A \to \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \mathbb{C}^r,$$

and the action of T is still linearized in $\bar{A}$. Tensoring $\bar{A}$ with a sufficiently ample line bundle on $\mathcal{J}^r \times C$ we obtain an ample line bundle on $\mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \mathbb{C}^r$, together with a linearization of the T-action in it, which has the additional property that the T-semi-stable locus is precisely $\mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \Omega$. Identifying $\mathscr{W} = p_*\big(\mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \mathbb{C}^r\big)$ with the space of morphisms from C into $\mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \mathbb{C}^r$ which represent the class of a section, we find ourselves in the situation studied in section 2 of [8], where is shown that in this case is possible to linearize (rather canonically) the T-action on $\mathscr{W}$ in an ample line bundle such that the corresponding semi-stable points have the property that their image is not completely contained in the unstable locus of $\mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \mathbb{C}^r$

(according to corollary 2.4 *loc.cit.*). Denoting $\mathscr{W}^s$ the set of T-semi-stable points of $\mathscr{W}$, we have found that $\mathscr{W}^s \subset \mathscr{W}^o$.

We want to prove now that $W = \mathscr{W}^s/T$ is projective; quasi-projectivity comes for free from the very construction, so that remains to prove the compactness (completeness). Since the T-action on $\mathscr{W}$ covers the identity of $\mathcal{J}^r$, $\mathscr{W}^s/T$ comes with the projection

$$q : \mathscr{W}^s/T \longrightarrow \mathcal{J}^r.$$

The first claim is that this map is surjective: indeed, according to theorem 2.5 in [8], points $s \in \mathscr{W}$ whose image is contained in $\mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \Omega$ are T-semi-stable. Since the 'bad' set $Z_X \subset \mathbb{C}^r$ has codimension at least two and because the line bundles $\mathcal{L}_\rho$ are globally generated, we deduce that for any $j \in \mathcal{J}^r$ there is a section $s \in \Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \mathbb{C}^r|_j)$ having the property that its image is disjoint from $\mathcal{P} \times_{(\mathbb{C}^*)^r} Z_X$. This proves the claim.

Consequently is enough to prove the compactness of the fibres of the projection q. The following lemma describes the fibrewise situation.

Lemma 2.3. *Suppose we are given a torus action on $\mathbb{C}^r$ as described at the beginning of section 1, so that the quotient is a smooth and projective toric variety. Let us consider now the action $T \times \mathbb{C}^R \to \mathbb{C}^R$, where*

$$\mathbb{C}^R := \mathbb{C}^{N_1} \oplus \cdots \oplus \mathbb{C}^{N_r}$$

and the torus T acts on the direct summands of $\mathbb{C}^R$ respectively by the same characters $(\chi_\rho)_\rho$ as on $\mathbb{C}^r$. Then the quotient is a smooth, compact toric variety.

Proof. The compactness of the quotient can be easily seen using the moment map description of toric varieties. In coordinates, the moment map corresponding to the T-action on $\mathbb{C}^r$ is (see section 3.3 in [5]):

$$m : \mathbb{C}^{n+l} \longrightarrow \mathbb{R}^l, \quad m(z) = \frac{1}{2}\begin{pmatrix} |z^1|^2 + a^1_\nu |z^{l+\nu}|^2 \\ \vdots \\ |z^l|^2 + a^l_\nu |z^{l+\nu}|^2 \end{pmatrix}. \tag{2.5}$$

Then X can be described as $X = m^{-1}(a)/T_{\mathbb{R}}$ for (a well chosen) $a \in \mathbb{R}^l$, where $T_{\mathbb{R}} = (S^1)^l$ denotes the real torus. Since X is compact, $m^{-1}(a)$ is still compact and $T_{\mathbb{R}}$ acts freely on it.

Let us move now to the new situation and denote $\underline{z}^\rho$ the points of $\mathbb{C}^{N_\rho}$. The moment map in this case has the form

$$\mathscr{M} : \mathbb{C}^R \longrightarrow \mathbb{R}^l, \quad \mathscr{M}(\underline{z}) = \frac{1}{2}\begin{pmatrix} |\underline{z}^1|^2 + a^1_\nu |\underline{z}^{l+\nu}|^2 \\ \vdots \\ |\underline{z}^l|^2 + a^l_\nu |\underline{z}^{l+\nu}|^2 \end{pmatrix} \tag{2.6}$$

and we want to prove that the quotient $Y := \mathscr{M}^{-1}(a)/T_{\mathbb{R}}$ is smooth and compact. It is easy to see that the maps

$$m^{-1}(a) \ni (z^1, \ldots, z^r) \longmapsto (|z^1|, \ldots, |z^r|) \in \mathbb{R}^r$$

and

$$\mathscr{M}^{-1}(a) \ni (\underline{z}^1, \ldots, \underline{z}^r) \longmapsto (|\underline{z}^1|, \ldots, |\underline{z}^r|) \in \mathbb{R}^r.$$

have the same image, which is compact since $m^{-1}(a)$ is so. The compactness of $\mathscr{M}^{-1}(a)$ is implied now by the compactness of standard spheres. The action of $T_{\mathbb{R}}$

on $\mathscr{M}^{-1}(a)$ is free because for any point $\underline{z}_o$ which solves the equation $\mathscr{M}(\underline{z}) = a$, is possible to find a $(\mathbb{C}^*)^r$-equivariant embedding $\mathbb{C}^r \hookrightarrow \mathbb{C}^R$ which pass through $\underline{z}_o$. The conclusion follows now from the fact that $T_{\mathbb{R}}$ acts freely on $m^{-1}(a)$. □

Coming back to our proposition, we deduce from the lemma that the fibres of q are smooth and compact toric varieties, all isomorphic to $Y = \mathscr{M}^{-1}(a)/T_{\mathbb{R}}$. Since $\mathscr{W}^s/T$ is quasi-projective, it follows that Y is actually projective and so is the invariant quotient $W = \mathscr{W}^s/T$.

Remains to prove that $\mathscr{W}^s = \mathscr{W}^o$: the natural inclusion $\mathscr{W}^s/T \hookrightarrow \mathscr{W}^o/T$ being an open map, its image is both open and closed in $\mathscr{W}^o/T$, so that $\mathscr{W}^s/T = \mathscr{W}^o/T$ and therefore $\mathscr{W}^s = \mathscr{W}^o$. □

According to (2.6), the moment map induced by the $T_{\mathbb{R}}$-action on $\mathscr{W}$ is given by

$$\mathscr{M}(s) = \frac{1}{2}\begin{pmatrix} \displaystyle\int_C \left(|s^1_\zeta|^2 + a^1_\nu |s^{l+\nu}_\zeta|^2\right)\mathrm{d}\gamma(\zeta) \\ \vdots \\ \displaystyle\int_C \left(|s^l_\zeta|^2 + a^l_\nu |s^{l+\nu}_\zeta|^2\right)\mathrm{d}\gamma(\zeta) \end{pmatrix} = \int_C m(s_\zeta)\mathrm{d}\gamma(\zeta), \tag{2.7}$$

where $s = (s^\rho)_\rho \in \mathscr{W}$ and $\mathrm{d}\gamma$ denotes a volume form on C. This formula is in agreement with the computations done in section 3 of [8], namely with equation (3.3) in there. Actually, this is the reason why we consider the level set $\{\mathscr{M} = a\} \subset \mathscr{W}$, and not another one, for describing the invariant quotient $\mathscr{W}/\!\!/T$.

We recall now that what we are actually interested in is a compactification of the space $\mathsf{Mor}_{\underline{d}}(C, X)$. The reason for introducing the variety W was to have a 'symmetric object' in our hands, in the sense that it does not depend on the choice of some particular cone of Σ. The compactification we are looking for is $V := \mathscr{V}/\!\!/T$ (see (2.2)), which can be now easily described as $V = \psi^* W = \mathfrak{J}^l \times_{\mathfrak{J}^r} W$. It is a fibre space over $\mathfrak{J}^l$, with all the fibres isomorphic to the toric variety Y constructed in lemma 2.3. We collect this information in the

Corollary 2.4. *The space of morphisms* $\mathsf{Mor}_{\underline{d}}(C, X)$ *is irreducible and the variety* $V := \psi^* W$ *is a smooth and projective compactification of it.*

Now becomes clear our statement in the introduction, that V is definitely not the stable map compactification of the space of morphisms from C to X, but is only birational to an irreducible component of this later. Indeed, the space of stable maps whose stabilized domain is C contains, when $g \geq 2$, the component whose points correspond to the following morphisms: the domain of definition is the singular curve consisting of C with $\mathbb{P}^1$ attached at some point; the map is constant on C and has multi-degree $\underline{d}$ on $\mathbb{P}^1$. Is also true that this component has strictly larger dimension than the expected one, and therefore is not clear how does it contribute to the Gromov-Witten invariants.

3. Cohomology of the space of morphisms

Since the projection $q : W \to \mathfrak{J}^r$ is a fibre bundle, the Leray-Hirsch theorem says that the cohomology of W is generated by the the cohomology of $\mathfrak{J}^r$ and the cohomology of the fibre Y (a similar remark applies to $q : V \to \mathfrak{J}^l$). Let us define now the line bundles

$$\Lambda_\rho := \mathscr{W}^o \times_{\chi_\rho} \mathbb{C} \longrightarrow W, \quad \forall \rho = 1, \ldots, r. \tag{3.1}$$

on W. The interest comes from the fact that there is a rational evaluation map

$$ev : V \times C \dashrightarrow X, \tag{3.2}$$

and the $\psi^*\Lambda_\rho$'s coincide respectively with the pull-backs under ev of the line bundles $\mathcal{O}(D_\rho)$ on X, at least on the domain of definition of ev. Since the classes D_ρ generate the cohomology of X, we may hope that integrals as $\int_V \prod_\rho (\psi^*\Lambda_\rho)^{m_\rho}$ are related to enumerative invariants of X. Morally, they should count the number of morphisms from C to X satisfying certain incidence conditions.

Proposition 3.1. *The integral cohomology of W is generated as a ring by the integral cohomology of $\mathcal{J}^r$ and the classes $\Lambda_1, \dots, \Lambda_r$.*

Proof. The statement follows from the fact that when restricted to the fibres of q, the classes $\Lambda_1, \dots, \Lambda_r$ generate the integral cohomology ring of Y (recall that Y is a toric variety). □

For making computations, we must find the relations among the classes Λ_ρ. The obvious relations are the linear ones like

$$\Lambda_{l+\nu} = a_\nu^\lambda \Lambda_\lambda, \quad \forall \nu \in \{1, \dots, n\}. \tag{3.3}$$

and the others corresponding to the remaining n-dimensional cones of Σ. In analogy with the case of toric varieties, we are going to describe the non-linear relations among the Λ_ρ's which arise from the primitive collections of Σ. We start noticing that the 'bad' sets which must be removed for constructing the quotients X and W behave in a rather functorial way: indeed, according to (2.4)

$$Z_W = \Gamma\big(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} Z_X\big) = \bigcup_\pi \Gamma\big(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \mathbb{A}(\pi)\big),$$

which is a union of linear subvector bundles of $\mathscr{W}$; as usual, $\pi \subset \{1, \dots, r\}$ runs over the primitive collections of the fan defining X. This means that the primitive collections of W, more precisely the primitive collections of the fan defining the toric fibre Y of $q : W \to \mathcal{J}^r$, are simply $\ker(\mathrm{pr}_\pi)$, for $\mathrm{pr}_\pi : \mathscr{W} \longrightarrow \oplus_{\rho\in\pi}\mathscr{W}_\rho$. We deduce that for any primitive collection π of the fan defining X,

$$\jmath_Y^*\left(\prod_{\rho\in\pi} \Lambda_\rho^{N_\rho}\right) = 0, \text{ for } N_\rho := \operatorname{rk}\mathscr{W}_\rho = d_\rho - (g-1). \tag{3.4}$$

However, we would like to have expressions for the products $\prod_{\rho\in\pi} \Lambda_\rho^{N_\rho}$ as elements of $H^*(W) \cong H^*(\mathcal{J}^r) \otimes H^*(Y)$. To begin with, we observe that for each ρ there is a *sheaf* monomorphism

$$0 \to \Lambda_\rho^{-1} \to q^*\mathscr{W}_\rho \quad \text{given by} \quad [s, z] \longmapsto z\,\mathrm{pr}_\rho\, s,\ \forall s \in \mathscr{W}^o \text{ and } \forall z \in \mathbb{C}.$$

Equivalently, one can say that for each ρ there is a canonical non-zero section $0 \to \mathcal{O}_W \to q^*\mathscr{W}_\rho \otimes \Lambda_\rho$. From the description of Z_W it follows that for every primitive collection π,

$$0 \longrightarrow \mathcal{O}_W \longrightarrow \bigoplus_{\rho\in\pi} q^*\mathscr{W}_\rho \otimes \Lambda_\rho \tag{3.5}$$

is a monomorphism of vector bundles, and consequently

Proposition 3.2. *For any primitive collection π of the fan defining X, the Euler class*

$$e\big(\oplus_{\rho\in\pi} q^*\mathscr{W}_\rho \otimes \Lambda_\rho\big) = 0.$$

The vanishing (3.4) is an immediate consequence, and when X is a projective space, so that $W = \mathbb{P}(\mathscr{W})$, proposition 3.2 reduces to the standard Grothendieck relation for $\mathcal{O}_{\mathbb{P}(\mathscr{W})}(1) \to W$. It is probably true that the linear relations (3.3) and the non-linear ones in proposition 3.2 generate the ideal of all the relations among the Λ_ρ's.

4. Localization

In this section we will apply the localization method developed in [2] for computing intersection products of the Λ_ρ's on W, the ultimate goal being to compute intersection numbers on V. We will apply the localization formula in cohomology with respect to the action of the torus S defined by the exact sequence (1.2).

First of all we have to make explicit the S-action on W and to describe the corresponding fixed point set W^S. Since Z_W defined by (2.4) is $(\mathbb{C}^*)^r$-invariant, its complement $\mathscr{W}^o$ is still $(\mathbb{C}^*)^r$-invariant and therefore $S = (\mathbb{C}^*)^r/T$ acts on $W = \mathscr{W}^o/T$ and moreover the evaluation map (3.2) is S-equivariant. This remark implies that if $[s] \in W^S$, then for all $\zeta \in C$ such that $s(\zeta) \notin Z_W$, $ev_{[s]}(\zeta) \in X^S$. But X^S consists of finitely many points: they correspond in a bijective fashion to the n-dimensional cones of Σ and their number equals the Euler characteristic of X. For $x \in X^S$, we shall denote $\sigma_n(x)$ the corresponding n-dimensional cone of Σ, and by $\bar{O}_x \subset \mathbb{C}^r$ the closure of the T-orbit above x. In fact $\bar{O}_x$ is the linear l-dimensional subspace of $\mathbb{C}^r$ defined by the equations

$$\bar{O}_x = \{z^\rho = 0 \mid \rho \in \sigma_n(x)\}.$$

Our discussion implies that for any $[s] \in W^S$ the image of the evaluation $ev_{[s]}$ is a point $x \in X^S$, and this in turn means that $s \in \Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \bar{O}_x)$. What we have obtained so far is that

$$W^S \subset \bigcup_{x \in X^S} \Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \bar{O}_x)^o/T =: \bigcup_{x \in X^S} W(x),$$

and our goal is to show that this inclusion is in fact an equality.

We are going to check that the component $\Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \bar{O}_{x_0})^o/T$ is fixed by S, for $x_0 \in X^S$ the point corresponding to the cone $\sigma_n(x_0) = \langle e_{l+1}, \dots, e_r \rangle$ of Σ. With respect to this choice of coordinates, the T-action on $\mathbb{C}^r$ is given by (1.4) and

$$[t_1, \dots, t_l, t_{l+1}, \dots, t_r] = [1, \dots, 1, \chi_1^{-1}(t')t_{l+1}, \dots \chi_l^{-1}(t')t_r] \text{ in } S,$$

for $t' = (t_1, \dots, t_l)$. Now is clear that any point $[s] \in W(x_0)$ is fixed because $\bar{O}_{x_0} = \{z^\rho = 0 \mid \rho = l+1, \dots, n\}$, so that $W(x_0) \subset W^S$. But we could have described the action of T on $\mathbb{C}^r$ using the coordinates furnished by any other n-dimensional cone of Σ and the conclusion would have been the same.

Proposition 4.1. *The fixed point set of the S-action on W is*

$$W^S = \bigcup_{x \in X^S} W(x), \text{ with } W(x) := \Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \bar{O}_x)^o/T.$$

Moreover, for any $x \in X^S$, $W(x)$ is defined by the fibre product

$$\begin{array}{ccc} W(x) & \longrightarrow & \prod_{\rho \notin \sigma_n(x)} \mathbb{P}(\mathscr{W}_\rho) \\ \downarrow & & \downarrow \\ \mathcal{J}^r & \xrightarrow{\mathrm{pr}_x} & \prod_{\rho \notin \sigma_n(x)} \mathcal{J}. \end{array}$$

Before proceeding we notice that the $W(x)$'s are smooth and disjoint subvarieties of W, which is in agreement with the general result obtained in [9].

Proof. The first part of the proposition being already proved, we are left with the second claim. Again, we are going to check it only for $x_0 \in X^S$, the general statement coming from the symmetry of the problem. We observe that $Z_X \cap \bar{O}_{x_0} = \bar{O}_{x_0} \setminus O_{x_0} = \bigcup_\lambda (\{z^\lambda = 0\} \cap \bar{O}_{x_0})$, because O_{x_0} is the locus where T acts freely. Therefore

$$\begin{aligned}\Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \bar{O}_{x_0})^o &= \Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \bar{O}_{x_0}) \setminus \Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} (Z_X \cap \bar{O}_{x_0})) \\ &= \textstyle\prod_{\lambda=1}^{l} \mathscr{W}_\lambda \setminus \prod_{\lambda=1}^{l} \{s \mid \mathrm{pr}_{\mathscr{W}_\lambda} s = 0\},\end{aligned}$$

and the statement follows because $T \cong (\mathbb{C}^*)^l$ acts componentwise. □

From the proposition we see that no matter what W looks like, its fixed point set for the torus action has a very down-to-earth description. A first byproduct is an explicit formula for the Euler number of the fibre Y.

Corollary 4.2.

$$\chi(Y) = \sum_{x \in X^S} \prod_{\rho \notin \sigma_n(x)} N_\rho.$$

Proof. This equality is just a rewriting of the main result in [9], but it can be proved in a more elementary way as follows: Y being a toric variety, its Euler characteristic coincides with the number fixed points under the $(\mathbb{C}^*)^R/T$-action. Since this big torus contains S, the fixed point set is contained in the union of the S-fixed subvarieties. But fibrewise these are just products of projective spaces on which $(\mathbb{C}^*)^R$ acts in standard fashion. □

For applying the localization formula we must know the action of S on the normal bundles to the fixed subvarieties.

Lemma 4.3. *For any $x \in X^S$, the normal bundle of the fixed component $W(x)$ of W fits in the following diagram:*

(4.1)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}(\mathcal{L}ie\, T) \arrow[r] \arrow[d, equal] & \bigoplus_{\rho \notin \sigma_n(x)} q^*\mathscr{W}_\rho \otimes \Lambda_\rho \arrow[r] \arrow[d] & \mathsf{T}_{W(x)/\mathcal{J}^r} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(\mathcal{L}ie\, T) \arrow[r] & \bigoplus_{\rho=1}^{r} q^*\mathscr{W}_\rho \otimes \Lambda_\rho \arrow[r] \arrow[d] & \mathsf{T}_{W/\mathcal{J}^r} \arrow[r] \arrow[d] & 0 \\
 & & \bigoplus_{\rho \in \sigma_n(x)} q^*\mathscr{W}_\rho \otimes \Lambda_\rho \arrow[r, "\cong"] \arrow[d] & \mathsf{N}_x := \mathsf{N}_{W(x)|W} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

For the trivial action of S on $\mathscr{W}_\rho$ and for the action

$$S \times \Lambda_\rho \longrightarrow \Lambda_\rho \text{ given by} \tag{4.2}$$
$$[t] \times [s,a] := [t \times s, t_\rho a], \quad \forall\, [t] \in S \text{ and } [s,a] \in \Lambda_\rho,$$

all the homomorphisms in the diagram above are S-equivariant.

Proof. Since $\mathscr{W}^o \to W$ is a principal T-bundle, we have the following S-equivariant exact sequence on W

$$0 \longrightarrow \mathcal{O}(\mathcal{L}ie\, T) \longrightarrow \mathsf{T}^{\mathrm{inv}}_{\mathscr{W}^o/\mathfrak{J}^r} \longrightarrow \mathsf{T}_{W/\mathfrak{J}^r} \longrightarrow 0, \tag{4.3}$$

where $\mathsf{T}^{\mathrm{inv}}_{\mathscr{W}^o/\mathfrak{J}^r}$ denotes the S-invariant relative tangent bundle to the total space of $\mathscr{W}^o$. But $\mathscr{W}^o$ is an open subset in a vector bundle over $\mathfrak{J}^r$, so that the relative tangent bundle is canonically isomorphic to $Q^*\mathscr{W} = \oplus_\rho Q^*\mathscr{W}_\rho$, for $Q : \mathscr{W} \to \mathfrak{J}^r$ the projection. As T preserves the decomposition of $\mathscr{W}$,

$$\mathsf{T}^{\mathrm{inv}}_{\mathscr{W}^o/\mathfrak{J}^r} \cong Q^*\mathscr{W}/T = \bigoplus_{\rho=1}^{r} Q^*\mathscr{W}_\rho/T.$$

We observe now that $Q^*\mathscr{W}_\rho/T \cong q^*\mathscr{W}_\rho \otimes \Lambda_\rho$, the isomorphism being given by

$$[s, w_\rho] \longmapsto w_\rho \otimes [s, 1]. \tag{4.4}$$

This proves the exactness of the middle row in the diagram (4.1). A similar argument proves the exactness of the first horizontal sequence, and the last row is now a simple consequence.

The very important thing which must be clarified yet is the way how S acts on $q^*\mathscr{W}_\rho \otimes \Lambda_\rho$. The sequence (4.3) being S-equivariant, we have to describe the induced action on $q^*\mathscr{W}_\rho \otimes \Lambda_\rho$ under the isomorphism (4.4). For $[t] \in S$,

$$[t] \times [s, w_\rho] = [t \times s, t_\rho w_\rho] \longmapsto t_\rho w_\rho \otimes [t \times s, 1] = w_\rho \otimes [t \times s, t_\rho],$$

so that we can see that indeed the S-action on $\mathscr{W}_\rho$ is trivial while the action on Λ_ρ is as in (4.2). $\square$

The next step is the computation of the equivariant first Chern classes for the restrictions of Λ_ρ to the fixed components $W(x)$. Before proceeding we notice that since $S = M^\vee \otimes_{\mathbb{Z}} \mathbb{C}^*$, there is a natural ring isomorphism $H^*(BS) \cong \mathrm{Sym}^\bullet M$, where BS denotes as usual the classifying space for S.

Lemma 4.4. For $x \in X^S$, denote $(u_\rho(x))_{\rho\in\sigma_n(x)} \subset M$ the dual basis to $(e^\rho)_{\rho\in\sigma_n(x)} \subset M^\vee$ formed by the integral generators of $\sigma_n(x)$. Then

$$c_1^S(\Lambda_\rho|_{W(x)}) = \Lambda_\rho|_{W(x)} + u_\rho(x), \quad \forall\, \rho \in \sigma_n(x),$$
$$c_1^S(\Lambda_\rho|_{W(x)}) = \Lambda_\rho|_{W(x)}, \quad \forall\, \rho \notin \sigma_n(x).$$

Proof. It is clear that $c_1^S(\Lambda_\rho|_{W(x)}) = \Lambda_\rho|_{W(x)} + u$, for some $u \in H^*(BS)$, and this element is precisely the weight of the action of S on the stalk $\Lambda_\rho|_{[s]}$ at some point $[s] \in W(x)$. Again, we shall make the computations for x_0 only: in this case the assignment $(\tau_{l+1}, \ldots, \tau_r) \mapsto [1, \ldots, 1, \tau_{l+1}, \ldots, \tau_r]$ gives an isomorphism $(\mathbb{C}^*)^n \xrightarrow{\cong} S$ and is easy to see that

$$(\mathbb{C}^*)^n \text{ acts on } \Lambda_\rho \begin{cases} \text{trivially, for } \rho = 1, \ldots, l \ \textit{i.e.}\ \rho \notin \sigma_n(x_0), \\ \text{by } \tau_\rho, \text{ for } \rho = l+1, \ldots, r \ \textit{i.e.}\ \rho \in \sigma_n(x_0). \end{cases}$$

A short computation shows that the above isomorphism is induced precisely by the choice of the dual basis to $(e^{l+1},\dots,e^r)$, and the conclusion follows. □

We are finally in position to apply the localization formula for computing intersection numbers. For positive integers $m_1,\dots,m_r$, we want is to compute the push-forward $q_*\big(\Lambda_1^{m_1}\cdot\ldots\cdot\Lambda_r^{m_r}\big)\in H^*(\mathcal{J}^r)$; for shorthand, we write ϕ for this product so that we must compute $q_*\phi$. The case of interest for us is when

$$m_1+\cdots+m_r=\dim V=N_1+\cdots+N_r+l(g-1).$$

We denote $\tilde{e}(\mathsf{N}_x)$ the equivariant Euler characteristic of $\mathsf{N}_x\to W(x)$ and let $\tilde{\phi}$ to be the equivariant analog of ϕ. If $\jmath_x: W(x)\hookrightarrow W$ is the inclusion, the localization formula for $\tilde{\phi}$ reads

$$\tilde{\phi}=\sum_{x\in X^S}(\jmath_x)_*\frac{\jmath_x^*\tilde{\phi}}{\tilde{e}(\mathsf{N}_x)}.$$

Since S acts trivially on $\mathcal{J}^r$, composing with the projection $q:W\to\mathcal{J}^r$ we obtain

$$q_*\tilde{\phi}=\sum_{x\in X^S}(q_x)_*\frac{\jmath_x^*\tilde{\phi}}{\tilde{e}(\mathsf{N}_x)}\in H_S^*(\mathcal{J}^r)=H^*(BS)\otimes H^*(\mathcal{J}^r), \tag{4.5}$$

for $q_x:=q\circ\jmath_x$, so that $q_*\phi$ is the $H^*(BS)$-free term in $q_*\tilde{\phi}$. Lemmas 4.3 and 4.4 imply that

$$\begin{aligned}\tilde{e}(\mathsf{N}_x)&=\prod_{\rho\in\sigma_n(x)}\Big((u_\rho(x)+\Lambda_\rho)^{N_\rho}-\theta_\rho(u_\rho(x)+\Lambda_\rho)^{N_\rho-1}+\frac{\theta_\rho^2}{2!}(u_\rho(x)+\Lambda_\rho)^{N_\rho-2}-\ldots\Big)\\&\overset{N_\rho>g}{=}\prod_{\rho\in\sigma_n(x)}(u_\rho(x)+\Lambda_\rho)^{N_\rho}\cdot\exp\Big(-\frac{\theta_\rho}{u_\rho(x)+\Lambda_\rho}\Big),\end{aligned}$$

where we are using the fact (see for instance page 336 in [1]) that the total Chern class of the Picard bundle $\mathscr{W}_\rho\to\mathcal{J}$ is $c(\mathscr{W}_\rho)=\exp(-\theta_\rho)$, with θ_ρ the class of the theta divisor (the lower index ρ indicates that we are on the ρ^{th} copy of $\mathcal{J}$ in $\mathcal{J}^r$). As a consequence,

$$\begin{aligned}\frac{\jmath_x^*\tilde{\phi}}{\tilde{e}(\mathsf{N}_x)}&=\frac{\displaystyle\prod_{\rho\notin\sigma_n(x)}\Lambda_\rho^{m_\rho}\cdot\prod_{\rho\in\sigma_n(x)}(u_\rho(x)+\Lambda_\rho)^{m_\rho}}{\displaystyle\prod_{\rho\in\sigma_n(x)}(u_\rho(x)+\Lambda_\rho)^{N_\rho}\cdot\exp\Big(-\frac{\theta_\rho}{u_\rho(x)+\Lambda_\rho}\Big)}\\&=\prod_{\rho\notin\sigma_n(x)}\Lambda_\rho^{m_\rho}\cdot\prod_{\rho\in\sigma_n(x)}(u_\rho(x)+\Lambda_\rho)^{m_\rho-N_\rho}\cdot\exp\Big(\frac{\theta_\rho}{u_\rho(x)+\Lambda_\rho}\Big).\end{aligned} \tag{4.6}$$

Plugging this into equality (4.5) we find the formula for the push forward of the class ϕ.

Theorem 4.5. For any positive integers m_ρ, $q_*\big(\Lambda_1^{m_1}\cdot\ldots\cdot\Lambda_r^{m_r}\big)$ is the constant term of the polynomial

$$\sum_{x\in X^S}(q_x)_*\Bigg[\prod_{\rho\notin\sigma_n(x)}\Lambda_\rho^{m_\rho}\cdot\prod_{\rho\in\sigma_n(x)}(u_\rho(x)+\Lambda_\rho)^{m_\rho-N_\rho}\cdot\exp\Big(\frac{\theta_\rho}{u_\rho(x)+\Lambda_\rho}\Big)\Bigg]. \tag{4.7}$$

It is certainly not apparent that this sum is a polynomial expression in the formal variables $u_\rho(x_0) =: u_\rho$, and computing its constant term is not an easy task (we must chose a basis of M for writing (4.7) as an element of $\mathrm{Sym}^\bullet M$, and $x_0 \in X$ is already our favorite fixed point in X). In most cases it is not true that the constant term of the whole sum is the sum of the individual constant terms. The question we are going to discuss is how to apply the result in concrete situations? Expanding the terms in (4.7) is a straightforward computation, but requires a little patience.

Lemma 4.6. *For $p \geq 0$,*

$$(u+\Lambda)^p \exp\left(\frac{\theta}{u+\Lambda}\right) = \sum_{0\le k\le p}\left[\sum_{0\le a\le k}\binom{p-a}{k-a}\Lambda^{k-a}\frac{\theta^a}{a!}\right]u^{p-k}$$
$$+\theta^{p+1}\sum_{0\le k}\left[\sum_{0\le b\le k}(-1)^{k-b}\binom{k}{k-b}\Lambda^{k-b}\frac{\theta^b}{(p+1+b)!}\right]\frac{1}{u^{k+1}},$$

while for $p \geq 1$,

$$\frac{1}{(u+\Lambda)^p}\exp\left(\frac{\theta}{u+\Lambda}\right) = \sum_{0\le k}\left[\sum_{0\le b\le k}(-1)^{k-b}\binom{p+k-1}{k-b}\Lambda^{k-b}\frac{\theta^b}{b!}\right]\frac{1}{u^{p+k}}.$$

The lemma implies that the expression (4.7) is a sum of rational functions, which are quotients of homogeneous polynomials in the $u_\rho(x)$'s. Its constant term is obtained by adding the functions having total degree zero (the total degree is the difference between the degrees of the nominator and the denominator).

For every $x \in X^S$, the coefficients of these functions are products in $\Lambda_\rho|_{W(x)}$, and their push-forward by q_x can be easily computed because, for $\rho \notin \sigma_n(x)$, the restriction of $\Lambda_\rho|_{W(x)}$ is simply (the pull-back of) the usual relatively ample line bundle $\mathcal{O}_{\mathbb{P}(\mathscr{W}_\rho)}(1) \to \mathbb{P}(\mathscr{W}_\rho)$ while the remaining line bundles $\Lambda_\rho|_{W(x)}$, $\rho \in \sigma_n(x)$, are linear combinations of the previous ones (see equation (3.3)).

The integrals which appear are of the type $(q_x)_*\big(\prod_{\rho\notin\sigma_n(x)}\Lambda_\rho^{k_\rho}\big)$, and is quite known that

(4.8)
$$(q_x)_*\left(\prod_{\rho\notin\sigma_n(x)}\Lambda_\rho^{k_\rho}\right) = \begin{cases}\displaystyle\prod_{\rho\notin\sigma_n(x)}\frac{\theta_\rho^{k_\rho-N_\rho+1}}{(k_\rho-N_\rho+1)!} & \text{if } N_\rho - 1 \le k_\rho \le N_\rho + g - 1\,\forall\rho,\\ 0 & \text{otherwise.}\end{cases}$$

Now we recall that we are actually interested in computing integrals on V, the compactification of the space $\mathsf{Mor}_{\underline{d}}(C,X)$, so that we must pull-back to $\mathfrak{J}^l$ the class $q_*\phi \in H^*(\mathfrak{J}^r)$, using the morphism ψ defined by (1.9). But ψ is explicitely given in terms of the combinatorics of the fan Σ.

The difficulty in applying formula (4.7) relies in the fact that we are not allowed to set to zero the variables $\{u_\rho(x)\}_{\rho,x}$. However, as we will see in a moment, for special choices of the exponents $(m_\rho)_\rho$ this is possible, and in this cases we obtain very explicit formulae for the corresponding intersection products.

Let us consider positive integers $a_1, \dots, a_r$ with the property that $a_1 + \dots + a_{r-1} - a_r = 0$ and take

$$\begin{aligned} m_\rho &= N_\rho + g + a_\rho = d_\rho + 1 + a_\rho, \text{ for } \rho = 1, \dots, r-1, \text{ and} \\ m_r &= N_r - (n-1)g - l - a_r = d_r - ng - (l + a_r - 1). \end{aligned} \tag{4.9}$$

Of course, such a choice is possible only when d_r is large enough for m_r to be positive. For such a choice, the total degree of the functions which appear in the products corresponding to the fixed points $x \in X^S$ with $r \in \sigma_n(x)$ is strictly negative; therefore these terms do not contribute to the intersection product. On the other hand, the products corresponding to $x \in X^S$ with $r \notin \sigma_n(x)$ are honest polynomials, so that we are allowed to set the variables to zero.

Proposition 4.7. *For integers $(m_\rho)_\rho$ as in* (4.9),

$$q_*(\Lambda_1^{m_1} \cdot \ldots \cdot \Lambda_r^{m_r}) = \sum_{x \in X^S,\, r \notin \sigma_n(x)} (q_x)_* \left[\prod_{\rho \notin \sigma_n(x)} \Lambda_\rho^{m_\rho} \cdot \prod_{\rho \in \sigma_n(x)} \left(\sum_{b=0}^{m_\rho - N_\rho} \Lambda_\rho^{m_\rho - N_\rho - b} \frac{\theta_\rho^b}{b!} \right) \right].$$

Analogous formulae can be obtained for any other $k \in \{1, \dots, r\}$ or, when it is possible, by taking several k's such that $m_k - N_k < 0$ in a suitable way.

We conclude this section with a vanishing result, which is a direct consequence of theorem 4.5.

Proposition 4.8. *Consider $J \subset \Sigma(1)$ such that the vectors $(e^\rho)_{\rho \in J}$ do not generate a cone of Σ (the primitive collections are the smallest subsets of $\Sigma(1)$ with this property). If $(m_\rho)_\rho$ are positive integers such that $m_\rho \geq N_\rho + g$ for all $\rho \in J$, then*

$$\int_V \Lambda_1^{m_1} \cdot \ldots \cdot \Lambda_r^{m_r} = 0.$$

Proof. Indeed, for such a set $\bigcap_{\rho \in J} D_\rho = \emptyset$ and consequently $J \setminus \sigma_n(x) \neq \emptyset$ for any $x \in X^S$ (otherwise $x \in \bigcap_{\rho \in J} D_\rho$, a contradiction). The conclusion follows now from the relations (4.8). □

We should point out that it is not always possible to chose integers with the property above: examples in this sense are the projective spaces. On the other hand, when there are primitive collections with less than $l = \operatorname{rank} T$ elements, there are integers having the desired property.

The conclusion of this article is that we were able to translate the integration problem on the space of morphisms from the curve C into the toric variety X in an integration problem on a power of the Picard torus of C. Interestingly enough, these integrals depend only on the combinatorics of the fan defining X and on the theta classes of the Picard varieties.

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Fakultät für Mathematik NA, Ruhr Universität Bochum, 44780 Bochum, Germany
E-mail address: halic@ag.ruhr-uni-bochum.de